Electricity and Magnetism, Exam 1, 22/02/2019

11 questions, 5 points each

Write your name and student number on the answer sheet. Use of a calculator is allowed. You may make use of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\mathbf{A}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector in the x-direction, and T is a scalar.

Differential calculus

1. Find the gradient of $f(x, y, z) = x^2y^3z^4$
   The gradient is

$$\nabla f(x, y, z) = \frac{\partial f_x}{\partial x}\hat{\mathbf{x}} + \frac{\partial f_y}{\partial y}\hat{\mathbf{y}} + \frac{\partial f_z}{\partial z}\hat{\mathbf{z}} = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$$

   +5 for correct answer, -2 when forgetting vector identity, -1 for minor calculation errors

2. Calculate the divergence of $\mathbf{v} = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$
   The divergence is

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = y + 2z + 3x$$

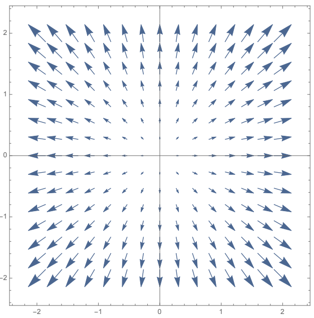
   +5 for correct answer, -2 when forgetting vector identity, -1 for minor calculation errors

3. Calculate the curl of $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$
   The curl is

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}) + \hat{\mathbf{y}}(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}) + \hat{\mathbf{z}}(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}) = -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}$$

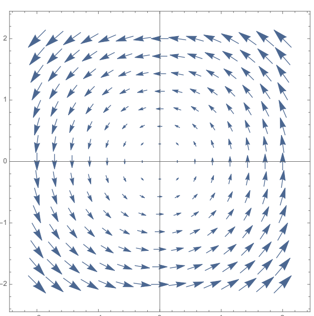
   +5 for correct answer, -2 when forgetting vector identity, -1 for minor calculation errors

4. Give an example of a two-dimensional vector field with **non-zero divergence and zero curl**. Provide the formula of the vector field, calculate the divergence and curl, and make a sketch of the field.
   A vector field whose magnitude is proportional to the coordinates would be a good example: $\mathbf{F} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. *For this field, the divergence is* $\nabla \cdot \mathbf{F} = 2$, *and the curl* $\nabla \times \mathbf{F} = 0$.



+5 for correct answer, +2 for formula, +2 for graph, +1 for divergence and curl calculated

5. Give an example of a two-dimensional vector field with **zero divergence and non-zero curl**. Provide the formula of the vector field, calculate the divergence and curl, and make a sketch of the field.
We need some rotation in the vector field, for example $\mathbf{F} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$. *The curl* $\nabla \times \mathbf{F} = 2\hat{\mathbf{z}}$, *and the divergence* $\nabla \cdot \mathbf{F} = 0$.



+5 for correct answer, +2 for formula, +2 for graph, +1 for divergence and curl calculated

6. Calculate the Laplacian $\nabla^2 T$ of the function $T = \sin(x)\sin(y)\cos(z)$.
Since $\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2} = -T$ *we find that* $\nabla^2 T = -3T$.
+5 for correct answer, +2 for formula / proper approach

7. Show how to calculate the volume of a sphere of radius R in spherical coordinates.

$$\begin{aligned} V = \int d\tau &= \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \; dr \; d\theta \; d\phi \\ &= \int_0^R r^2 \; dr \int_0^{\pi} \sin\theta \; d\theta \; \int_0^{2\pi} d\phi \\ &= \frac{R^3}{3} \; 2 \; 2\pi = \frac{4}{3}\pi R^3 \end{aligned}$$

+5 correct answer, +2 correct formula, +1 correct boundary, +2 correct execution

8. $\int_{-\infty}^{\infty} x^3 \delta(3 - x) dx =$
For $x = 3$ *the delta function is non-zero; therefore the answer is* $3^3 = 27$.
+5 correct answer, -2 for forgetting cube power

9. Suppose $\mathbf{v} = \frac{2}{r^2}\hat{\mathbf{r}}$. The divergence $\boldsymbol{\nabla} \cdot \mathbf{v} =$
The answer is $8\pi\delta^3(\mathbf{r})$
+5 correct answer, -1 for forgetting factor 2, 2 for 8π only answer, 0 points for $\nabla \cdot \mathbf{v} = 0$

10. Show, using Gauss's law, that the electric field outside a uniformly charged solid sphere of radius R and total charge q is the same as it would have been if all charge had been concentrated at the center.
This can be found using a spherical surface with radius $r > R$ as a Gaussian surface. Since with this surface the electric field $\mathbf{E}$ is always parallel to the surface normals, the dot product from $\int \mathbf{E} \cdot d\mathbf{a}$ can be dropped, and we get $\int |\mathbf{E}| da = |\mathbf{E}| 4\pi r^2$, which through Gauss's law is equal to the enclosed charge, and we find
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}},$$
which is just the field of a charge q located at the origin.
+5 correct answer, +3 for gauss's law + correct surface, +2 correct execution, -1 forgetting $\hat{\mathbf{r}}$

11. An infinite plane carries a uniform surface charge σ per unit area . Find its electric field.
Use a Gaussian pillbox, to find that $Q_{enc} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry $\mathbf{E}$ points away from the surface, and we find a contribution from both top and bottom surfaces: $\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}| = \frac{1}{\epsilon}_0 \sigma A$. We know the direction of the field; it points away from the surface, so
$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$
+5 correct answer, +3 gauss's law + correct surface, -1 forgetting $\hat{\mathbf{n}}$, -1 forgetting $\frac{1}{2}$, 2 if only answer is given

The End